

Quantum implementation of the cosmic censorship conjecture for toroidal black holes

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Abstract

We consider some aspects of quantum field theory of a conformally coupled scalar field on the singular background obtained in the massless limit of a class of toroidal black holes. The stress-tensor and its back reaction on the metric are computed using point-splitting regularization, in the cases of transparent, Neumann and Dirichlet boundary conditions. We find that the quantum fluctuations generate an event horizon which hides the singularity. The resulting object can be interpreted as a long lived remnant. We discuss the relevance of this result in the context of the cosmic censorship conjecture, and in connection to the end point of the quantum evaporation process.

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I. INTRODUCTION

Recently, interest has grown about topological black holes [1–5], namely solutions of the vacuum Einstein equations with a negative cosmological constant, which are asymptotically anti-de Sitter black holes, and have a compact event horizon with a nontrivial topology. These black holes can result from gravitational collapse [6] and from quantum tunnelling processes [7]. They can be charged [8,5], exist in dilaton gravity [8,9], and also admit a rotating generalization [10,11]. The reason why they violate the classical theorems of general relativity forbidding nonspherical topologies is the presence of the cosmological constant. For instance, the class of toroidal black holes is described by the metric

$$ds^2 = - \left(-\frac{2\eta}{r} + \frac{r^2}{\ell^2} \right) dt^2 + \left(-\frac{2\eta}{r} + \frac{r^2}{\ell^2} \right)^{-1} dr^2 + r^2 (|\tau|^2 dx^2 + dy^2 + 2\text{Re}\tau dx dy) \quad (1)$$

where $x, y \in [0, 1[$ and τ is the Teichmüller complex parameter that determines the conformal class of the torus. If $\eta > 0$ the solution (1) is a black hole of mass $M = \eta |\text{Im } \tau| / 4\pi$ with an event horizon localized in $r_+ = (2\eta\ell^2)^{1/3}$. As the mass decreases to zero the event horizon shrinks and disappears for $\eta = 0$, leaving a naked singularity. We want to investigate this singular background and check on its stability against the formation of an event horizon when quantum fluctuations are taken into account in a semiclassical approach. In particular we shall study the propagation of a conformally coupled scalar field on this background and evaluate the back reaction on the metric, for Dirichlet, Neumann and transparent boundary conditions, and show that the quantum fluctuations of the field induce the formation of an event horizon which hides the singularity.

The cosmic censorship conjecture states that, during gravitational collapse, naked singularities never occur, at least for asymptotically flat spacetimes [12,13]. However this conjecture is violated in presence of nontrivial topology; Lemos has constructed a solution of collapsing matter with toroidal symmetry that leads to the background (1) with $\eta = 0$, hence to a naked singularity [14]. It is interesting to investigate how quantum mechanics modifies the issue of the collapse; we wonder if quantum fluctuations, which cannot be neglected in the final stages of the collapse, can prevent the classical naked singularities to form, hiding them behind an event horizon. If this shows to be the case, semiclassical back reaction of quantum fields on the geometry would strengthen the cosmic censorship conjecture. An instance of this is the case of extreme black holes, where quantum emission drives the hole away from extremality [15]. The issue of these questions is also fundamental in the problem of the end point of evaporation. Black holes are known to evaporate by the emission of thermal radiation [16]. In the case of toroidal topology, the black hole would loose mass and end the evaporation process leaving a naked singularity. Semiclassical results showing a dressing of the final singularity would indicate that the final state could be a remnant, or perhaps that the evaporation does not stop at $\eta = 0$, but would continue through a serie of dressed conical spacetimes, as has been suggested in three dimensions [17]. However, in the final stages of evaporation, quantum effects are expected to dominate, thus no final conclusion can be drawn in this sense without a full theory of quantum gravity.

Analogous results have been obtained by various authors [18,17,19,20] for the non-rotating Bañados-Teitelboim-Zanelli (BTZ) black hole [21,22]. This three-dimensional topological black hole shows the same behavior: in the $M \rightarrow 0$ limit, the space-time acquires a

naked singularity, but quantum fluctuations produce a horizon at the semiclassical level. We shall extend those results to the four-dimensional case, which is physically more relevant. The analogy with the BTZ black hole is also formal: the background (1) with $\eta = 0$ is the quotient of the four-dimensional anti-de Sitter space with respect to a discrete group of isometries. This fact simplifies the computation of the quantum stress-tensor of a conformally coupled scalar field, and leads to interpret it as a consequence of Casimir's effect.

II. THE GEOMETRY OF THE PROBLEM

In this section we shall discuss the geometry of the background \mathcal{M} of toroidal black holes, described by the singular metric

$$ds^2 = -\frac{r^2}{\ell^2} dt^2 + \frac{\ell^2}{r^2} dr^2 + r^2 (dX^2 + dY^2) \quad (2)$$

where t ranges over \mathbb{R} , $r > 0$ and $X, Y \in [0, 1[$ with the points 0 and 1 identified. A first important remark about this metric is that it is a locally anti-de Sitter (AdS) metric and can be obtained quotienting AdS space with respect to a discrete subgroup of its isometry group.

Let $\mathcal{V} = \mathbb{R}_2^3$ be the flat pseudo-Euclidean space with coordinates (x, y, z, u, v) endowed with the metric $\eta_{ab} = (1, 1, 1, -1, -1)$ that defines the scalar product $(\cdot, \cdot)_{\mathcal{V}}$. Anti-de Sitter space is defined to be the hypersurface [23]

$$AdS_4 = \{\mathbf{x} \in \mathcal{V} \mid \eta_{ab} x^a x^a = -\ell^2\} \quad (3)$$

endowed with the induced metric. To avoid closed timelike curves, we shall work with the universal covering \overline{AdS} of anti-de Sitter space. By construction, \overline{AdS} is a homogeneous and symmetric manifold, with isometry group $SO(3, 2)$. It solves Einstein equations with negative cosmological constant $\Lambda = -3/\ell^2$ and has a constant negative curvature $R = -12/\ell^2$.

Let us parametrize \overline{AdS} space by the coordinates (t, r, X, Y) defined by

$$t = -\frac{\ell v}{u+x}, \quad r = u+x, \quad X = \frac{y}{u+x}, \quad Y = \frac{z}{u+x}. \quad (4)$$

The metric assumes the form (2) in these coordinates, and is the universal covering of the considered manifold. To obtain the background \mathcal{M} we need to make the identifications $X \sim X + m$ and $Y \sim Y + n$ with $m, n \in \mathbb{Z}$. The manifold \mathcal{M} is regular everywhere except for $r = 0$. This is not a curvature singularity, but there are inextendible geodesics ending there as the elements of the identification group leave this point fixed. These identifications form an abelian subgroup $H \cong \mathbb{Z} \times \mathbb{Z} \subset SO(3, 2)$ of the homogeneity group. In the \mathcal{V} space the elements of the identification group are represented by the matrices

$$(\Lambda^m)^a{}_b = \begin{pmatrix} 1 - \frac{m^2}{2} & -m & 0 & -\frac{m^2}{2} & 0 \\ m & 1 & 0 & m & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \frac{m^2}{2} & m & 0 & 1 + \frac{m^2}{2} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (\Gamma^n)^a{}_b = \begin{pmatrix} 1 - \frac{n^2}{2} & 0 & -n & -\frac{n^2}{2} & 0 \\ 0 & 1 & 0 & 0 & 0 \\ n & 0 & 1 & n & 0 \\ \frac{n^2}{2} & 0 & n & 1 + \frac{n^2}{2} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

corresponding respectively to the $X \mapsto X + m$ and the $Y \mapsto Y + n$ translations. It can be proven that these matrices satisfy the abelian properties $\Lambda^m \Lambda^n = \Lambda^{m+n}$, $\Gamma^m \Gamma^n = \Gamma^{m+n}$, and $\Lambda^m \Gamma^n = \Gamma^n \Lambda^m$, and therefore we deduce that the manifold \mathcal{M} is the quotient $\overline{\text{AdS}}/H$.

III. GREEN FUNCTIONS

Now we want to study the propagation of a conformally coupled scalar field ϕ on the background \mathcal{M} and calculate its Green function. The total action of the theory is $S = S_g + S_m$, where S_g is the usual Hilbert-Einstein action for the gravitational field

$$S_g[g_{\mu\nu}] = \frac{1}{2\kappa} \int \sqrt{-g} (R - 2\Lambda) \quad (5)$$

with $\kappa = 8\pi G$, and S_m is the action of the matter field

$$S_m[g_{\mu\nu}, \phi] = -\frac{1}{2} \int \sqrt{-g} [g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi + \xi R \phi^2] \quad (6)$$

with $\xi = 1/6$. The equations of motion are as usual $G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}$ and $(\square - \xi R) \phi = 0$.

As \mathcal{M} is a quotient of $\overline{\text{AdS}}$, its Green function can be obtained from that of the covering space $\overline{\text{AdS}}$ by the method of images. But $\overline{\text{AdS}}$ is not a globally hyperbolic spacetime, therefore we have to impose boundary conditions at infinity. To this end we conformally map $\overline{\text{AdS}}$ to half of the Einstein static universe (ESU). The boundary conditions at infinity become conditions on the fields on the equator of ESU. As shown in [24], three choices of boundary conditions are possible; the transparent boundary conditions are obtained quantizing the field using smooth modes over the whole ESU, while the Neumann and Dirichlet boundary conditions can be imposed on the equator of the ESU. The Green functions are then

$$\mathcal{G}_{\text{AdS}}^\epsilon(\mathbf{x}, \mathbf{x}') = \frac{1}{4\pi^2} \frac{1}{(\mathbf{x} - \mathbf{x}', \mathbf{x} - \mathbf{x}')_\nu} + \frac{\epsilon}{4\pi^2} \frac{1}{(\mathbf{x} + \mathbf{x}', \mathbf{x} + \mathbf{x}')_\nu} \quad (7)$$

with $\epsilon = 0, +1, -1$ for transparent, Neumann and Dirichlet boundary conditions respectively. The functional form of the Green functions is the same as in Minkowski space because the background is conformally flat and the field conformally coupled. The second term in Eq. (7) corresponds to an antipodal image in $\overline{\text{AdS}}$ needed to force the boundary conditions. The same results can be obtained by a direct mode sum [25,26].

Now we obtain the Green function with the appropriate boundary conditions simply by summing over images, getting

$$\mathcal{G}_{\mathcal{M}}^\epsilon(\mathbf{x}, \mathbf{x}') = \frac{1}{4\pi^2} \sum_{m,n \in \mathbb{Z}} \left[\frac{1}{(\mathbf{x} - \Lambda^m \Gamma^n \mathbf{x}', \mathbf{x} - \Lambda^m \Gamma^n \mathbf{x}')_\nu} + \frac{\epsilon}{(\mathbf{x} + \Lambda^m \Gamma^n \mathbf{x}', \mathbf{x} + \Lambda^m \Gamma^n \mathbf{x}')_\nu} \right]. \quad (8)$$

The series converges in the Dirichlet case, but diverges like $\sum [1/(m^2 + n^2)]$ for transparent and Neumann boundary conditions [17]. This is an infrared divergence, due to the existence of zero modes in Neumann and transparent cases, that render the Klein-Gordon operator non-invertible. Hence, we have to regularize the Green functions eliminating these zero modes. This can be done defining the regularized Green functions by

$$\mathcal{G}_{\mathcal{M}}^{\epsilon}(\mathbf{x}, \mathbf{x}'|s) = \frac{1}{4\pi^2} \sum_{m,n \in \mathbb{Z}} \left[\frac{1}{(\mathbf{x} - \Lambda^m \Gamma^n \mathbf{x}', \mathbf{x} - \Lambda^m \Gamma^n \mathbf{x}')_{\mathcal{V}}^s} + \frac{\epsilon}{(\mathbf{x} + \Lambda^m \Gamma^n \mathbf{x}', \mathbf{x} + \Lambda^m \Gamma^n \mathbf{x}')_{\mathcal{V}}^s} \right] \quad (9)$$

which are convergent for $s > 1$ and diverge in the $s \rightarrow 1$ limit. Note that Eq. (9) is a Green function in $(2s + 2)$ -dimensional anti-de Sitter space: we have used dimensional regularization. In the calculation of the stress tensor we shall take derivatives of these functions with respect to the spacetime variables, that will render finite the series in $s = 1$. In this point, the stress tensor will be the analytic continuation of the dimensionally continued stress-tensor.

IV. RENORMALIZED STRESS TENSOR

The stress tensor for the field ϕ is given by

$$\begin{aligned} T_{\mu\nu} &\equiv \frac{-2}{\sqrt{-g}} \frac{\delta S_m}{\delta g^{\mu\nu}} \\ &= \frac{2}{3} \nabla_{\mu} \phi \nabla_{\nu} \phi - \frac{1}{3} \phi \nabla_{\mu} \nabla_{\nu} \phi - \frac{1}{6} g_{\mu\nu} g^{\alpha\beta} \nabla_{\alpha} \phi \nabla_{\beta} \phi + \frac{1}{3} g_{\mu\nu} \phi \square \phi + \frac{1}{2\ell^2} g_{\mu\nu} \phi^2. \end{aligned} \quad (10)$$

We observe first that the tensor $T_{\mu\nu}$ is traceless, as is expected in a classical conformally invariant theory. However, as we are in even dimension, a conformal anomaly appears and breaks this invariance at the quantum level.

Now, we aim to calculate the quantum stress tensor of the theory. We cannot take the expectation value of formula (10) directly: after the quantization procedure the field $\phi(\mathbf{x})$ becomes a distribution and expectation values like $\langle 0 | \phi(\mathbf{x}) \phi(\mathbf{x}) | 0 \rangle$ are formally divergent quantities; we need to define a regularization procedure. We shall use the point-splitting technique [27]; in this framework, we define the regularized stress tensor to be

$$\begin{aligned} T_{\mu\nu}(\mathbf{x}) &= \lim_{\mathbf{x}' \rightarrow \mathbf{x}} \frac{1}{3} \left[\nabla_{\mu} \nabla'_{\nu} + \nabla'_{\mu} \nabla_{\nu} - \frac{1}{2} \nabla_{\mu} \nabla_{\nu} - \frac{1}{2} \nabla'_{\mu} \nabla'_{\nu} \right. \\ &\quad \left. - \frac{1}{2} g_{\mu\nu} \left(g^{\alpha\beta} \nabla_{\alpha} \nabla'_{\beta} + \frac{1}{\ell^2} \right) \right] [\mathcal{G}(\mathbf{x}, \mathbf{x}') - \mathcal{G}_{\text{sing}}(\mathbf{x}, \mathbf{x}')], \end{aligned} \quad (11)$$

where $\mathcal{G}_{\text{sing}}(\mathbf{x}, \mathbf{x}')$ is a singular part of the Green function, and we have used the equations of motion.

We have to distinguish two contributions to the Green function: the former is the direct contribution, that is the Green function of $\overline{\text{AdS}}$ space, the latter is the contribution coming from the sum over the images. The $\overline{\text{AdS}}$ contribution is divergent in the limit $\mathbf{x}' \rightarrow \mathbf{x}$ and we have to subtract its singular part to renormalize it. In fact, the stress tensor of the scalar field in $\overline{\text{AdS}}$ has to be proportional to the only tensor available, i.e., the metric tensor $g_{\mu\nu}$. The proportionality factor is one quarter of the conformal anomaly in our space, which is found to be [27]

$$\langle T^{\mu}_{\mu} \rangle = \frac{\hbar}{2880\pi^2} (R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} - R_{\mu\nu} R^{\mu\nu} - \square R) = -\frac{4\hbar}{960\pi^2 \ell^4}, \quad (12)$$

from which we deduce that the stress tensor contribution of $\overline{\text{AdS}}$ amounts to

$$\langle T_{\mu\nu}^{AdS} \rangle = -\frac{\hbar}{960\pi^2\ell^4} g_{\mu\nu}. \quad (13)$$

In contrast, the contribution of the images remains finite after the coincidence limit of the Green function. A calculation of the image contribution $\langle T_{\mu\nu}^\pm \rangle$ in the \mathcal{V} space followed by a projection on \mathcal{M} [18] yields

$$\langle T_{\mu\nu}^\pm(\mathbf{x}) \rangle = \frac{\hbar}{6\pi^2} \sum'_{m,n \in \mathbb{Z}} \left(S_{\mu\nu}^{\pm(m,n)}(\mathbf{x}) - \frac{1}{4} g_{\mu\nu} g^{\alpha\beta} S_{\alpha\beta}^{\pm(m,n)}(\mathbf{x}) \right), \quad (14)$$

where by $T_{\mu\nu}^\pm$ we mean respectively the stress tensor calculated on the Green functions $(1/4\pi^2)(\mathbf{x} \pm \mathbf{x}', \mathbf{x} \pm \mathbf{x}')_{\mathcal{V}}^{-s}$, the prime on the sum indicates that we drop the $(m, n) = (0, 0)$ term and $S_{\mu\nu}^{\pm(m,n)}$ is the pull-back $S_{\mu\nu}^{\pm(m,n)} = (\partial_\mu \mathbf{x}^a)(\partial_\nu \mathbf{x}^b) S_{ab}^{\pm(m,n)}$ of the tensor in $T_{\mathbf{x}}^* \mathcal{V}$

$$\begin{aligned} S_{ab}^{\pm(m,n)}(\mathbf{x}|s) = & 2s(s+1) \frac{(\Lambda^m \Gamma^n)_a{}^c x_c (\Lambda^{-m} \Gamma^{-n})_b{}^d x_d + (\Lambda^{-m} \Gamma^{-n})_a{}^c x_c (\Lambda^m \Gamma^n)_b{}^d x_d}{|x \pm \Lambda^m \Gamma^n x|^{2(s+2)}} \\ & - s(s+1) \frac{(\Lambda^m \Gamma^n)_a{}^c x_c (\Lambda^m \Gamma^n)_b{}^d x_d + (\Lambda^{-m} \Gamma^{-n})_a{}^c x_c (\Lambda^{-m} \Gamma^{-n})_b{}^d x_d}{|x \pm \Lambda^m \Gamma^n x|^{2(s+2)}} \\ & \mp s \frac{(\Lambda^m \Gamma^n)_{ab} + (\Lambda^m \Gamma^n)_{ba}}{|x \pm \Lambda^m \Gamma^n x|^{2(s+1)}}. \end{aligned} \quad (15)$$

Now the series (14) is convergent in the point $s = 1$, hence we can safely remove the cutoff by setting $s = 1$. Finally, the image contribution to the stress tensor is given by Eq. (14), with

$$\begin{aligned} S_{ab}^{\pm(m,n)}(\mathbf{x}) = & 4 \frac{(\Lambda^m \Gamma^n)_a{}^c x_c (\Lambda^{-m} \Gamma^{-n})_b{}^d x_d + (\Lambda^{-m} \Gamma^{-n})_a{}^c x_c (\Lambda^m \Gamma^n)_b{}^d x_d}{|x \pm \Lambda^m \Gamma^n x|^6} \\ & - 2 \frac{(\Lambda^m \Gamma^n)_a{}^c x_c (\Lambda^m \Gamma^n)_b{}^d x_d + (\Lambda^{-m} \Gamma^{-n})_a{}^c x_c (\Lambda^{-m} \Gamma^{-n})_b{}^d x_d}{|x \pm \Lambda^m \Gamma^n x|^6} \\ & \mp \frac{(\Lambda^m \Gamma^n)_{ab} + (\Lambda^m \Gamma^n)_{ba}}{|x \pm \Lambda^m \Gamma^n x|^4}. \end{aligned} \quad (16)$$

The total stress tensors with the various boundary conditions considered read

$$\langle T_{\mu\nu}^\epsilon \rangle = \langle T_{\mu\nu}^+ \rangle + \epsilon \langle T_{\mu\nu}^- \rangle + \langle T_{\mu\nu}^{AdS} \rangle. \quad (17)$$

Writing Eq. (14) in the coordinates (t, r, X, Y) and performing the pull-back we obtain

$$\langle T_{\mu\nu}^+ \rangle = \frac{2\hbar r^4}{3\pi^2} \sum'_{m,n \in \mathbb{Z}} \frac{1}{(4\ell^2 + (m^2 + n^2)r^2)^3} \text{diag}((m^2 + n^2)\ell^{-2}; 0; 2m^2 - n^2; 2n^2 - m^2)$$

and

$$\langle T_{\mu\nu}^- \rangle = \frac{\hbar}{2\pi^2 r^2} \sum'_{m,n \in \mathbb{Z}} \frac{1}{(m^2 + n^2)^3} \text{diag}\left(-(m^2 + n^2)\ell^{-2}; (m^2 + n^2)\frac{\ell^2}{r^4}; n^2 - 3m^2; m^2 - 3n^2\right).$$

From now on we define

$$\alpha \equiv \sum'_{m,n \in \mathbb{Z}} \frac{1}{(m^2 + n^2)^2} = \sum'_{m,n \in \mathbb{Z}} \frac{3m^2 - n^2}{(m^2 + n^2)^3} = 6.0268 \dots \quad (18)$$

Written in terms of the dimensionless variable $\zeta = r/2\ell$, the stress-tensor reads

$$\begin{aligned} \langle T_{\mu\nu}^\epsilon \rangle = \frac{\hbar}{2\pi^2\ell^2} \text{diag} & \left(-\frac{\alpha}{r^2} + \frac{\epsilon}{3\ell^2} \left(\frac{r}{2\ell} \right)^4 E \left(\frac{r}{2\ell} \right); \frac{\alpha\ell^4}{r^6}; -\frac{\alpha}{r^2}\ell^2 + \frac{\epsilon}{6} \left(\frac{r}{2\ell} \right)^4 E \left(\frac{r}{2\ell} \right); \right. \\ & \left. -\frac{\alpha}{r^2}\ell^2 + \frac{\epsilon}{6} \left(\frac{r}{2\ell} \right)^4 E \left(\frac{r}{2\ell} \right) \right) + \langle T_{\mu\nu}^{AdS} \rangle \end{aligned} \quad (19)$$

where we have defined the auxiliary function (the derivative of an Epstein function)

$$E(\zeta) \equiv \sum_{m,n \in \mathbb{Z}} \frac{m^2 + n^2}{(1 + (m^2 + n^2)\zeta^2)^3}. \quad (20)$$

The stress tensor we have obtained is regular and finite except for $r = 0$ where it diverges. Furthermore, as can be easily verified, it is conserved. Finally, we notice that the found stress tensor involves a negative energy density, that may be interpreted as a Casimir energy in \mathcal{M} due to the identifications under the group H .

V. BACK REACTION ON THE METRIC

Now, we want to compute the semiclassical corrections to the geometry of \mathcal{M} at order \hbar due to the presence of the quantum field ϕ . These corrections arise by solving the semiclassical equations

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa \langle T_{\mu\nu} \rangle. \quad (21)$$

Let us first consider first the $\overline{\text{AdS}}$ term of the stress tensor. As $\langle T_{\mu\nu}^{AdS} \rangle$ is proportional to the metric, it only contributes to change the cosmological constant by a finite renormalization: namely,

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = -\frac{\hbar\kappa}{960\pi^2\ell^4} g_{\mu\nu} \quad \Leftrightarrow \quad G_{\mu\nu} + \Lambda_{ren} g_{\mu\nu} = 0 \quad (22)$$

where

$$\Lambda_{ren} = \Lambda + \frac{\hbar\kappa}{960\pi^2\ell^4} = -\frac{3}{\ell^2} + \frac{\hbar\kappa}{960\pi^2\ell^4}. \quad (23)$$

Having performed this renormalization we can forget the $\overline{\text{AdS}}$ contribution in the stress tensor and consider only the contribution of the images to the corrections to the metric. In the following we take ℓ to be the renormalized constant.

We look now for a static solution, with toroidal symmetry, of the semiclassical equations. The general metric possessing these properties has the form

$$ds^2 = -f^2(r) dt^2 + f^{-2}(r) dr^2 + e^{2A(r)} (dX^2 + dY^2). \quad (24)$$

The semiclassical equations become

$$\begin{cases} -f^2 [2f^2 A'' + (f^2)' A' + 3f^2 (A')^2 + \Lambda] = \kappa \langle T_{tt} \rangle \\ f^{-2} [(f^2)' A' + f^2 (A')^2 + \Lambda] = \kappa \langle T_{rr} \rangle \\ e^{2A} \left[\frac{1}{2} (f^2)'' + (f^2)' A' + f^2 A'' + f^2 (A')^2 + \Lambda \right] = \kappa \langle T_{XX} \rangle = \kappa \langle T_{YY} \rangle \end{cases} \quad (25)$$

As we want to study the corrections of order \hbar to the metric of \mathcal{M} we define

$$f^2(r) = \frac{r^2}{\ell^2} + \hbar \bar{f}(r) + \mathcal{O}(\hbar^2), \quad A(r) = \ln r + \hbar \bar{A}(r) + \mathcal{O}(\hbar^2) \quad (26)$$

and substitute them in Eq. (25). The 0-order terms satisfy the equations and cancel out, leaving, at first order, the linear equations

$$\begin{cases} -\frac{\hbar r^2}{\ell^2} \left[\frac{1}{r} \bar{f}' + \frac{1}{r^2} \bar{f} + 2 \frac{r^2}{\ell^2} \bar{A}'' + 8 \frac{r}{\ell^2} \bar{A}' \right] = \kappa \langle T_{tt} \rangle \\ \frac{\hbar \ell^2}{r^2} \left[\frac{1}{r} \bar{f}' + \frac{1}{r^2} \bar{f} + 4 \frac{r}{\ell^2} \bar{A}' \right] = \kappa \langle T_{rr} \rangle \\ \hbar r^2 \left[\frac{1}{2} \bar{f}'' + \frac{1}{r} \bar{f}' + \frac{r^2}{\ell^2} \bar{A}'' + 4 \frac{r}{\ell^2} \bar{A}' \right] = \kappa \langle T_{XX} \rangle = \kappa \langle T_{YY} \rangle. \end{cases} \quad (27)$$

A linear combination of the first two equations yields

$$\frac{d}{dr} [r^2 \bar{A}'] = -\frac{\kappa \ell^2}{2\hbar} \left[\frac{\ell^2}{r^2} \langle T_{tt} \rangle + \frac{r^2}{\ell^2} \langle T_{rr} \rangle \right] \quad (28)$$

which, integrated twice, gives

$$\bar{A}'(r) = -\frac{\epsilon \kappa}{12\pi^2 (2\ell)^3} \left(\frac{2\ell}{r} \right)^2 E_1 \left(\frac{r}{2\ell} \right), \quad \bar{A}(r) = -\frac{\epsilon \kappa}{12\pi^2 (2\ell)^2} E_2 \left(\frac{r}{2\ell} \right) \quad (29)$$

where we have defined

$$E_1(\zeta) \equiv \int \zeta^2 E(\zeta) d\zeta, \quad E_2(\zeta) \equiv \int \frac{1}{\zeta^2} E_1(\zeta) d\zeta. \quad (30)$$

We do not need the exact form of $\bar{A}(r)$; but it can be proven that $\exp(2A)$ behaves asymptotically like r^2 and is regular everywhere. Substituting \bar{A} in the Einstein equations we find the correction to the lapse function

$$\bar{f}(r) = -\frac{\alpha \kappa}{2\pi^2 r^2} + \frac{\epsilon \kappa}{3\pi^2 \ell^2} \left(\frac{2\ell}{r} \right) E_3 \left(\frac{r}{2\ell} \right) \quad (31)$$

with

$$E_3(\zeta) \equiv \int \zeta E_1(\zeta) d\zeta. \quad (32)$$

Performing the integrations and subtracting an infinite integration constant we find

$$E_3(\zeta) = \sum_{m,n \in \mathbb{Z}} ' \left[\left(\frac{\zeta^2}{16(m^2 + n^2)^{1/2}} - \frac{3}{16(m^2 + n^2)^{3/2}} \right) \left(\arctan(\sqrt{m^2 + n^2}\zeta) - \frac{\pi}{2} \right) + \frac{\zeta}{16(m^2 + n^2)} + \frac{\zeta}{8(m^2 + n^2)(1 + (m^2 + n^2)\zeta^2)} \right] + \eta. \quad (33)$$

The asymptotic behavior of this function is

$$E_3(\zeta) = \frac{\alpha}{3\zeta} + \eta + \mathcal{O}\left(\frac{1}{\zeta^3}\right), \quad (34)$$

from which we deduce the asymptotic form of the lapse function

$$\begin{cases} f^2(r) = \frac{r^2}{\ell^2} - \frac{1}{18} \frac{\hbar \kappa \alpha}{\pi^2 r^2} + \frac{\hbar \eta}{r} + \mathcal{O}\left(\frac{\hbar}{r^4}\right) & (N) \\ f^2(r) = \frac{r^2}{\ell^2} - \frac{17}{18} \frac{\hbar \kappa \alpha}{\pi^2 r^2} + \frac{\hbar \eta}{r} + \mathcal{O}\left(\frac{\hbar}{r^4}\right) & (D) \end{cases} \quad (35)$$

for Neumann and Dirichlet boundary conditions respectively. Hence, the integration constant η corresponds to the physical mass of the black hole. It is not fixed by the semiclassical theory, but we can set it equal to zero, to match asymptotically the corrected metric with its classical background, and to obtain a massless solution.

Thus the semiclassical metric acquires a curvature singularity in the origin and the perturbation due to the propagation of the quantum field ϕ develops a horizon that hides the singularity.

This is easily seen in the case of transparent boundary conditions ($\epsilon = 0$), where a horizon forms in $r_+ = (\kappa \alpha \hbar \ell^2)^{1/4}$. In terms of the Planck length ℓ_p the black hole radius goes like $\sqrt{\ell \ell_p}$ and formally has a Hawking temperature [16]

$$T = \frac{\hbar r_+}{\pi \ell^2}, \quad (36)$$

which is immensely smaller than the temperature of the initially macroscopic toroidal black hole. If the dressed singularity continues to evaporate, then it is a kind of long lived remnant. When Neumann or Dirichlet boundary conditions are imposed, the expression of the lapse function is more complicated, and we cannot find analytically the location of the horizon. However, it is still possible to prove that in the perturbed space-time the singularity is hidden behind an event horizon. We start from the fact that $E_3(\zeta)$ is decreasing for positive ζ and assumes in 0 the value

$$E_3(0) = \frac{3\pi}{32} \alpha_{3/2} = 2.66 \dots \quad \alpha_{3/2} \equiv \sum_{m,n \in \mathbb{Z}} ' (m^2 + n^2)^{-\frac{3}{2}} = 9.033 \dots \quad (37)$$

Thus, we obtain the inequality

$$0 < E_3(\zeta) \leq \frac{3\pi}{32}\alpha_{3/2}, \quad \zeta \geq 0. \quad (38)$$

Substituting it in the expression (35), we find that the lapse function (for Neumann and Dirichlet boundary conditions respectively) is restricted to vary between two simple functions

$$\begin{cases} \frac{r^2}{\ell^2} - \frac{\hbar\kappa\alpha}{2\pi^2 r^2} \leq f^2(r) \leq \frac{r^2}{\ell^2} - \frac{\hbar\kappa\alpha}{2\pi^2 r^2} + \frac{\hbar\kappa\alpha_{3/2}}{16\pi\ell r} & (N) \\ \frac{r^2}{\ell^2} - \frac{\hbar\kappa\alpha}{2\pi^2 r^2} - \frac{\hbar\kappa\alpha_{3/2}}{16\pi\ell r} \leq f^2(r) \leq \frac{r^2}{\ell^2} - \frac{\hbar\kappa\alpha}{2\pi^2 r^2} & (D) \end{cases} \quad (39)$$

that are negative for sufficiently small r and become positive as r grows. This is more evident if we multiply Eq. (39) by $\ell^2 r^2$:

$$\begin{cases} r^4 - \frac{\hbar\kappa\alpha\ell^2}{2\pi^2} \leq \ell^2 r^2 f^2(r) \leq r^4 - \frac{\hbar\kappa\alpha\ell^2}{2\pi^2} + \frac{\hbar\kappa\alpha_{3/2}\ell}{16\pi} r & (N) \\ r^4 - \frac{\hbar\kappa\alpha\ell^2}{2\pi^2} - \frac{\hbar\kappa\alpha_{3/2}\ell}{16\pi} r \leq \ell^2 r^2 f^2(r) \leq r^4 - \frac{\hbar\kappa\alpha\ell^2}{2\pi^2} & (D). \end{cases} \quad (40)$$

Since a quartic equation of the form $x^4 + ax - b = 0$, with $b > 0$, has a unique real and positive root, inspection into Eqs. (40) shows that the lapse function is zero in r_+^N (r_+^D) for Neumann (Dirichlet) boundary conditions, negative for lower values of r and positive for higher values. Furthermore, the zeroes of the lapse function satisfy $r_+^D > r_+ > r_+^N$, and, as ℓ grows, r_+^D and r_+^N approach r_+ . Thus, the quantum fluctuations dress the naked singularity. For a Neumann field we find the horizon nearer to the singularity, while Dirichlet boundary conditions push the horizon outwards, and transparent boundary conditions give rise to an horizon located between the other two.

VI. CONCLUSIONS

In this paper we have analyzed the propagation of a conformally coupled scalar field on the singular manifold \mathcal{M} . The main result is that the quantum fluctuations dress the singularity. We have argued that this quantum dressing could be a general feature of propagation of quantum fields on singular backgrounds generated by collapsing matter, and would imply that quantum effects would work in the sense of the cosmic censorship conjecture. Furthermore, as \mathcal{M} appears to be the natural end point of the evaporation process, the semiclassical insight seems to discredit the possibility of naked singularities as end point of evaporation. However, as the horizon enjoys thermal properties, the best we can conclude is that the dressed singularity behaves as a kind of long lived remnant.

A further extension of these results could be the inclusion of the Teichmüller complex parameter, which specifies conformally equivalent classes of the torus, and see if it introduces significant changes; another interesting question could be the generalization of the present discussion to other kinds of field, like conformal Weyl spinors fields or vector fields.

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